

THE GENERIC RANK FOR A -PLANNAR STRUCTURES.

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ABSTRACT. The paper mostly collects material on generic rank of A -modules with respect to differential geometric applications. Our research was motivated by geometry of A -structures. In particular, we discuss the case where A is an unitary associative algebra not necessary with inversion. Some of the examples are studied in detail.

1. MOTIVATION

Let us say a few words about our geometric motivation. Various concepts generalizing geodetics have been studied for almost quaternionic and similar geometries. Also various structures on manifolds are defined as smooth distribution in the vector bundle $T^*M \otimes TM$ of all endomorphisms of the tangent bundle. Very well known are two examples: almost complex and almost quaternionic structures. Let us extract some formal properties from these examples. Unless otherwise stated, all manifolds are smooth and they have the dimension m . Let ∇ be a linear connection and let $c : \mathbb{R} \rightarrow M$ be a curve on M . Then there is a vector field $\dot{c} := \frac{\partial c(t)}{\partial t} : \mathbb{R} \rightarrow TM$ along the curve c . Classically, a curve c is a geodesic if and only if its tangent vectors $\dot{c}(t)$ are parallely transported along $c(t)$. Let M be a smooth manifold equipped with a linear connection ∇ and let F be an affinor on M . A curve c is called F -planar curve if there is its parametrization $c : \mathbb{R} \rightarrow M$ satisfying the condition

$$\nabla_{\dot{c}} \dot{c} \in \langle \dot{c}, F(\dot{c}) \rangle.$$

It is easy to see that geodesics are F -planar curves for all affinors F , because of $\nabla_{\dot{c}} \dot{c} \in \langle \dot{c} \rangle \subset \langle \dot{c}, F(\dot{c}) \rangle$. The best known example is an almost complex structure. We have to be careful about the dimension of M . Let M be a manifold of dimension two and let I be a complex structure. A curve c is F -planar for $F = I$ if and only if c is satisfying the identity $\nabla_{\dot{c}} \dot{c} \in \langle \dot{c}, I\dot{c} \rangle \cong \mathbb{R}^2$, and any curve c satisfy the identity $\nabla_{\dot{c}} \dot{c} \in \mathbb{R}^2$. In other words any curve c is F -planar on the manifold of dimension two. The concept of F -planar curves makes sense for dimension at least four. Consider almost hypercomplex structure (I, J, K) . The curve $c : \mathbb{R} \rightarrow M$ such that $\nabla_{\dot{c}} \dot{c} \in \langle \dot{c}, I(\dot{c}), J(\dot{c}), K(\dot{c}) \rangle$ is called 4-planar. It is easy to see that all geodesics are 4-planar curves, because of $\nabla_{\dot{c}} \dot{c} \in \langle \dot{c} \rangle \subset \langle \dot{c}, I(\dot{c}), J(\dot{c}), K(\dot{c}) \rangle$ and also all F -planar curves are 4-planar, if $F \in \langle I, J, K, E \rangle$. This simple consequence of standard behavior of the generators of a vector subspace suggests the generalization of the planarity concept below.

Definition. 1.1. Let M be a smooth manifold of dimension m . Let A be a smooth ℓ -rank ($\ell < m$) vector subbundle in $T^*M \otimes TM$, such that the identity

1991 *Mathematics Subject Classification.* 53B10.

Key words and phrases. Linear connection, Geodesics, F-planar, A-planar, Almost complex structure, Projective structure, Distributions, Cliffordian structure .

affinor $E = id_{TM}$ restricted to $T_x M$ belongs to $A_x \subset T_x^* M \otimes T_x M$ at each point $x \in M$. We say that M is equipped by ℓ -rank A -structure.

In Definition 1.1, the dimension of M is higher than the rank of A . This is not a restriction, because there are no A -structures of rank ℓ higher than m . The possibility $\ell = m$ is not interesting, because in this event every curve is A -planar.

Definition. 1.2. For any tangent vector $X \in T_x M$ we shall write $A(X)$ for the vector subspace

$$A(X) = \{F(X) | F \in A_x M\} \subset T_x M$$

and we call $A(X)$ the A -hull of the vector X . Similarly, the A -hull of vector field will be subbundle in TM obtained pointwise.

2. THE GENERIC RANK

For every smooth parameterized curve $c : \mathbb{R} \rightarrow M$ we write \dot{c} and $A(\dot{c})$ for the tangent vector field and its A -hull along the curve c .

Definition. 2.1. Let M be a smooth manifold equipped with an A -structure and a linear connection ∇ . A smooth curve $c : \mathbb{R} \rightarrow M$ is told to be A -planar if

$$\nabla_{\dot{c}} \dot{c} \in A(\dot{c}).$$

Clearly, A -planarity means that the parallel transport of any tangent vector to c has to stay within the A -hull $A(\dot{c})$ of the tangent vector field \dot{c} along the curve.

Definition. 2.2. Let (M, A) be a smooth manifold M equipped with an ℓ -rank A -structure. We say that the A -structure has

- (1) *generic rank* ℓ if for each $x \in M$ the subset of vectors $(X, Y) \in T_x M \oplus T_x M$, such that the A -hulls $A(X)$ and $A(Y)$ generate a vector subspace $A(X) \oplus A(Y)$ of dimension 2ℓ is open and dense in $T_x M \oplus T_x M$.
- (2) *weak generic rank* ℓ if for each $x \in M$ the subset of vectors

$$\mathcal{V} := \{X \in T_x M | \dim A(X) = \ell\}$$

is open and dense in $T_x M$.

One immediately checks that any A -structure which has generic rank ℓ has weak generic rank ℓ . Indeed, if $U \subset T_x M$ is an open subset of vectors X with $A(X)$ of dimension lower than ℓ , then $U \times U$ is an open subset with too low dimension, too.

Lemma. 2.3. Let M be a smooth manifold of dimension at least two and F be an affinor such that $F \neq q \cdot E$, $q \in \mathbb{R}$. Then the A -structure, where $A = \langle E, F \rangle$ has weak generic rank 2.

Proof. Consider A -structure $A = \langle E, F \rangle$. The complement of \mathcal{V} consists of vectors $X \in T_x M$ such that:

$$X + aF(X) = 0, \quad a \in \mathbb{R},$$

i.e. eigenspace of F . Dimension of A is two and F is not multiple of the identity. Thus, the union of eigenspaces of F is closed or trivial vector subspace of $T_x M$. Thus, the complement \mathcal{V} is open and nontrivial, i.e. open and dense. \square

There is only one possibility for the A -structures in the lowest dimension one $A = \langle E \rangle$. The algebra $\langle E \rangle$ is an algebra with inversion, such that $E \cdot E = E$. For every $X \in T_x M$, $A(X)$ is the straight line containing A .

Finally, let us recall two important examples. The pair (M, F) , where M is a smooth manifold and F is an affinor on M is called a complex structure if and only if $F^2 = -E = -id_M$. An almost complex structure has generic rank two on all manifolds of dimension at least four, because of Lemma 2.3. The pair (M, F) is called a product structure on M if and only if $F^2 = E$ and $F \neq E$. An almost product structure has generic rank two on all manifolds of dimension at least four, because of Lemma 2.3.

3. THE CASE WHERE A IS AN ALGEBRA

Lemma. 3.1. *Every A -structure (M, A) on a manifold M , $\dim M \geq \dim A$, where A is an algebra with inversion, has weak generic rank $\dim A$.*

Proof. Consider X such that $X \notin \mathcal{V}$, therefore $\exists F \in A = \langle E, G \rangle$, $FX = 0$, and $F^{-1}FX = 0$ implies $X = 0$. \square

Theorem. 3.2. *Let M be an A -structure and let X_1, \dots, X_m be a basis of $V := T_x M$, i.e. V is an A -module. Let A be an n -dimensional k -algebra, where $n < m$. If there exists $X \in V$ such that $\dim(A(X)) = n$ then the A -structure has weak generic rank n .*

Proof. We prove equivalent statement, A -module V does not have a generic rank ℓ if and only if there is a vector $X \in V$, such that for any vector $Y \in V$ there is an affinor G_Y , such that $G_Y(X - \epsilon Y) = 0$, for small ϵ . Therefore, for any vector $Y \in V$ there is an affinor G_Y such that

$$G_Y(X) = \epsilon G_Y(Y)$$

for small ϵ . Hence, the affinor $\frac{1}{\sqrt{\epsilon}}G_Y$ maps $\frac{1}{\sqrt{\epsilon}}X$ to a vector $G_Y(Y)$ and therefore, for any vector $Y \in V$ there is an affinor H_Y such that $H_Y(\frac{1}{\sqrt{\epsilon}}X) = H_Y(Y)$. In particular, there is an affinor S such that $S(\frac{1}{\sqrt{\epsilon}}X) = S(Y + \frac{1}{\sqrt{\epsilon}}X)$ and therefore, for any $Y \in V$ there is an affinor S_Y such that $S_Y(Y) = 0$. \square

Theorem. 3.3. *Let (M, A) be a smooth manifold of dimension m equipped with A -structure of rank ℓ , such that $2\ell \leq m$. If A_x is an algebra (i.e. for all $f, g \in A_x$, $fg := f \circ g \in A_x$) for all $x \in M$, and A has weak generic rank ℓ then the structure has generic rank ℓ .*

Proof. Since the A -structure has a weak generic rank ℓ , there is an open and dense subset $\mathcal{V} \subset TM$ such that $\dim A(X) = \ell$ for all $X \in \mathcal{V}$.

Because A is an algebra, for any $X, Z \in TM$, $Z \in A(X)$ implies also $A(Z) \subset A(X)$, and moreover $A(Z) = A(X)$ for all $X, Z \in \mathcal{V}$ because of the dimension. Thus, whenever there is a non-trivial vector $0 \neq Z \in A(X) \cap A(Y)$, the entire subspaces coincide, i.e. $A(X) = A(Y)$.

In particular, whenever $X, Y \in \mathcal{V}$ and the dimension of $A(X) + A(Y)$ is less than 2ℓ , we know $A(X) = A(Y)$.

Let us consider a couple of vectors $(Y, Z) \in A(X) \oplus A(X)$ for some $X \in \mathcal{V}$. Consider a vector $W \notin A(X)$. An open neighbourhood \mathcal{U} of Y has to include $(Y + aW, Y)$ for all sufficiently small $a \in \mathbb{R}$. But if $Y + aW \in A(X)$ for some

$a \neq 0$ then $W \in A(X)$ and this is not true. Thus, for every couple of vectors in $A(X) \oplus A(X)$ and for its every open neighbourhood, we have found another couple $(Y' = Y + aW, Z)$ for which the dimension of $A(Y') + A(Z)$ is 2ℓ . This proves the density of the set of couples of vectors generating the maximal dimension 2ℓ .

Of course, the requirement on the maximal dimension is an open condition and the theorem is proved. \square

Corollary. 3.4. *Let (M, A) be a smooth manifold with A -structure of rank ℓ , such that $2\ell \leq \dim M$. If $A_x \subset T_x^*M \otimes T_xM$ is an algebra with inversion then A has weak generic rank. Moreover, if $\dim M \geq 2\ell$ then A has generic rank ℓ .*

Corollary. 3.5. *Let (M, A) be a smooth manifold with A -structure of rank ℓ , such that $2\ell \leq \dim M$ and A is an algebra. If there exists $X \in T_xM$ such that $\dim(A(X)) = n$ then the A -structure has generic rank n .*

4. REMARK ON FROBENIUS ALGEBRAS

Let A be an algebra over \mathbb{R} with basis $\{F_i\}$, where $i = 1, \dots, n$, $F_1 := E$, with structure constants

$$C_{ij}^k \in \mathbb{R} \text{ (i.e. } F_i F_j = C_{ij}^s F_s \text{)}.$$

In particular, one can easily see that

$$F_i = F_1 F_i = C_{1i}^s F_s = \delta_i^s F_s, \text{ i.e. } C_{1i}^s = \delta_i^s.$$

We introduce matrices

$$\hat{C}_i = (C_{ji}^k), \quad \hat{C}_i^* = (C_{ik}^j),$$

where j is a number of rows. Then the associativity condition can be written as

$$\hat{C}_j \hat{C}_k = C_{jk}^s \hat{C}_s \text{ or } \hat{C}_j^* \hat{C}_k^* = C_{jk}^s \hat{C}_s^*$$

and unity can be written as $\hat{C}_1 = E$. A linear functional $\epsilon : A \rightarrow \mathbb{R}$ is determined by the choice of a n -dimensional vector $\lambda = (\lambda_1, \dots, \lambda_n)$.

Now, for $\epsilon(F_i) = \lambda_i$ and for $F = \sum_{i=1}^n a_i F_i \in A$ we can see immediately that $\epsilon(F) = \sum_{i=1}^n a_i \lambda_i$ and finally $F_i F_j = C_{ij}^s F_s$. We prove that, if $\lambda \in \mathbb{R}^n$ be a vector such that the matrix $G = (g_{ij})$ is regular, where $g_{ij} := C_{ij}^s \lambda_s$. The functional $\epsilon : A \rightarrow \mathbb{R}$, such that

$$\epsilon : \sum_{i=1}^n a_i F_i \mapsto \sum_{i=1}^n a_i \lambda_i$$

is a Frobenius form.

The formula for generic rank n from the Theorem 3.2. reads that if there exists $X \in V$ such that $\{F_i X\}$ are linearly independent then V has a generic rank. On the other hand, if there exists $\lambda \in \mathbb{R}^n$ such that $\{\hat{C}_i \lambda\}$ are linearly independent then A is a Frobenius algebra. This indicates that these properties lead to similar conditions.

In other words, if A is an algebra over \mathbb{R} and the matrices \hat{C}_i are structural matrices of A , then there is B -module \mathbb{R}^n , where $B = \langle \hat{C}_1, \dots, \hat{C}_n \rangle$. Therefore, the algebra A is a Frobenius algebra if and only if the B -module \mathbb{R}^n has generic rank n .

5. EXAMPLES

One can apply these results to two big groups of geometric structures, Clifford algebras and distributions.

5.1. An almost Cliffordian manifolds. An almost Clifford and almost Cliffordian manifolds are G -structures based on the definition of Clifford numbers. An almost Clifford manifold based on $Cl(s, t)$ is given by a reduction of the structure group $GL(km, \mathbb{R})$ to $GL(m, \mathcal{O})$, where $k = 2^{s+t}$, $m \in \mathbb{N}$ and \mathcal{O} is an arbitrary Clifford algebra. An almost Cliffordian manifold is given by a reduction of the structure group to $GL(m, \mathcal{O})GL(1, \mathcal{O})$. It is easy to see that an almost Cliffordian structure is an A -structure, where A is a Clifford algebra \mathcal{O} because the affinors in the form of $F_0, \dots, F_\ell \in A$ can be defined only locally. In [4] authors prove the following theorem.

Theorem. 5.2. *Let F_0, \dots, F_k denote the $k+1$ elements of the matrix representation of Clifford algebra $Cl(s, t)$. Then there exists a real vector X such that the dimension of a linear span $\langle F_i X | i = 1, \dots, k \rangle$ equals to $k+1$.*

Finally, let M be a Cliffordian manifold, i.e. let (M, A) be a smooth manifold with $A = Cl(s, t)$, such that $2^{s+t+1} \leq \dim M$, then Cliffordian manifold has generic rank 2^{s+t} . For more information about almost Cliffordian structures see papers [4, 6]

5.3. Distributions. If D, \bar{D} form a complete system of distributions (i.e. they are disjoint and $D + \bar{D} = TM$) than there are two affinors P, \bar{P} associate with them such that

$$P^2 = P, \bar{P}^2 = \bar{P}, P\bar{P} = \bar{P}P = 0 \text{ and } P + \bar{P} = E,$$

where $\text{rank } P = r$ and $\text{rank } \bar{P} = \bar{r}$.

The representation of distributions by affinors can be extended to any complete system D_i such that the affinors P_i satisfy the properties

$$P_i^2 = P_i, P_i P_j = 0 \text{ for } i \neq j, \text{ and } \sum_i P_i = E.$$

Considering the element $P = a_1 P_1 + \dots + a_n P_n \in A$, the matrix

$$\begin{pmatrix} EP \\ P_1 P \\ \vdots \\ P_n P \end{pmatrix}$$

is following

$$\begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 \dots & 0 & a_n \end{pmatrix}$$

and therefore $\langle P_1, \dots, P_n \rangle$ has weak generic rank n by Lemma 3.2. Finally, let M be a manifold with complete system of distributions D_1, \dots, D_n , i.e. let (M, A) be a smooth manifold with $A = \langle P_1, \dots, P_n \rangle$, such that $2n \leq \dim M$, then the A -structure has generic rank n .

ACKNOWLEDGMENT

This work was supported by the European Regional Development Fund in the IT4Innovations Center of Excellence project CZ.1.05/1.1.00/02.0070.

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